A Theory of Economic Slack

Pascal Michaillat

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APPENDIX H.

Convexity of the Phillips curve

In this appendix, we establish that the Phillips curve $\pi(u)$ is strictly convex in u under the assumption that the matching function is Cobb-Douglas and symmetric:

$$m(u, v) = \mu\sqrt{uv}.$$

It is difficult to obtain more general analytical results, so for different matching functions, numerical analysis is required, as we do in chapter 14.

H.1. Symmetric Cobb-Douglas matching function

We assume in this appendix that the matching function is Cobb-Douglas with elasticity $\sigma = 1/2$. This means that the trading rates are

$$f(\theta) = \mu\sqrt{\theta} \quad \text{and} \quad q(\theta) = \frac{\mu}{\sqrt{\theta}}. \tag{H.1}$$

The matching elasticity is constant at $\eta(\theta) = \sigma = 1/2$, so

$$\frac{\eta(\theta)}{1 - \eta(\theta)} = 1. \tag{H.2}$$

The unemployment rate is given by (8.8), so

$$u(\theta) = \frac{\lambda}{\lambda + \mu\sqrt{\theta}}. \tag{H.3}$$

Inverting this strictly decreasing relationship, we write tightness as a strictly decreasing function of the unemployment rate:

$$\theta(u) = \left[\frac{\lambda(1-u)}{\mu u}\right]^2. \tag{H.4}$$

The highest possible value for tightness is $\theta \to \bar{\theta}$, when $\tau(\theta) \to \infty$. The upper bound $\bar{\theta}$ is defined such that $q(\bar{\theta}) = \kappa\lambda$, so here

$$\bar{\theta} = \left(\frac{\mu}{\kappa\lambda}\right)^2. \tag{H.5}$$

The lowest possible unemployment rate is achieved when $\theta \to \bar{\theta}$. We denote this minimum unemployment rate by $\bar{u} = u(\bar{\theta})$; it is given by

$$\bar{u} = \frac{\kappa\lambda^2}{\mu^2 + \kappa\lambda^2}. \tag{H.6}$$

H.2. Preliminary result

RESULT H.1. *The ratio between the matching wedge and unemployment rate admits the closed form*

$$\frac{\tau(u)}{u} = \frac{1-\bar{u}}{u-\bar{u}} - \frac{1}{u}. \tag{H.7}$$

The ratio is strictly decreasing and strictly convex in u on $(\bar{u}, 1)$.

The proof has three steps: derive the closed form (H.7), then sign the first and second derivatives.

H.2.1. Closed form for $\tau(u)/u$

Combining (H.1) and (H.4) allows us to express the ad-filling rate as a function of the unemployment rate:

$$q(u) = \frac{\mu^2}{\lambda} \cdot \frac{u}{1-u}.$$

Using expression (8.13), we can express the matching wedge as a function of the unemployment rate too:

$$\tau(u) = \frac{\kappa\lambda}{\frac{\mu^2}{\lambda}\frac{u}{1-u} - \kappa\lambda} = \frac{\kappa\lambda^2(1-u)}{\mu^2 u - \kappa\lambda^2(1-u)} = \frac{\kappa\lambda^2(1-u)}{(\mu^2+\kappa\lambda^2)u - \kappa\lambda^2} = \frac{\bar{u}(1-u)}{u-\bar{u}}.$$

Then, we find that the ratio between matching wedge and unemployment rate is a simple function of the unemployment rate:

$$\frac{\tau(u)}{u} = \frac{\bar{u}(1-u)}{u(u-\bar{u})} = \frac{\bar{u} - u + u - \bar{u}u}{u(u-\bar{u})} = \frac{1-\bar{u}}{u-\bar{u}} - \frac{1}{u},$$

which establishes (H.7).

H.2.2. First derivative of $\tau(u)/u$

Next, we compute the first derivative of the ratio τ/u. We know that τ is increasing in tightness, while the unemployment rate is decreasing in tightness, so τ/u is increasing in tightness, which means that it must be decreasing in the unemployment rate. So we know that the first derivative must be negative. But the expression for the first derivative will be useful to establish that the ratio is convex in the unemployment rate.

Differentiating (H.7) once with respect to the unemployment rate, we get

$$\frac{d\tau/u}{du} = \frac{1}{u^2} - \frac{1-\bar{u}}{(u-\bar{u})^2}. \tag{H.8}$$

We can quickly verify that the derivative is strictly negative. The sign of the derivative is the sign of

$$\begin{aligned}(u-\bar{u})^2 - (1-\bar{u})u^2 &= u^2 + \bar{u}^2 - 2u\bar{u} - u^2 + \bar{u}u^2 \\ &= \bar{u}\left[\bar{u} - 2u + u^2\right] \\ &= \bar{u}\left[(\bar{u} - u) + u(u-1)\right].\end{aligned}$$

Since $0 < \bar{u} < u < 1$, we have $u - 1 < 0$ and $\bar{u} - u < 0$, so we verify that the first derivative is strictly negative.

H.2.3. Second derivative of $\tau(u)/u$

Differentiating (H.8) with respect to the unemployment rate, we now get the second derivative of the ratio:

$$\frac{d^2\tau/u}{du^2} = \frac{2(1-\bar{u})}{(u-\bar{u})^3} - \frac{2}{u^3}.$$

The sign of the second derivative is the sign of

$$\begin{aligned}(1-\bar{u})u^3 - (u-\bar{u})^3 &= u^3 - \bar{u}u^3 - u^3 + \bar{u}^3 + 3u^2\bar{u} - 3u\bar{u}^2 \\ &= \bar{u}\left[-u^3 + \bar{u}^2 + 3u^2 - 3u\bar{u}\right] \\ &= \bar{u}\left[u^2 - u^3 + \bar{u}^2 + u^2 - 2u\bar{u} + u^2 - u\bar{u}\right]\end{aligned}$$

$$= \bar{u}\left[u^2(1-u) + (\bar{u}-u)^2 + u(u-\bar{u})\right].$$

Since $0 < \bar{u} < u < 1$, all the terms in the square bracket are positive: $u^2(1-u) > 0$, $(\bar{u}-u)^2 > 0$, and $u(u-\bar{u}) > 0$. Hence the second derivative is strictly positive, which implies that $\tau(u)/u$ is strictly convex in u.

H.3. Main result

We now pass from convexity of $\tau(u)/u$ to convexity of the equilibrium Phillips curve in an unemployment-inflation plane. The equilibrium Phillips curve $\pi(\theta)$ is given by (14.20), where here (H.2) holds. We compose it with the function $\theta(u)$ given by (H.4) to obtain a Phillips curve linking inflation to unemployment:

$$\pi(u) = \pi^* + \frac{1}{\varpi\delta(\delta - r)a} \cdot y(u) \cdot \left[\frac{\tau(u)}{u} - 1\right].$$

The ratio $\tau(u)/u$ is given by (H.7) and output is given by $y(u) = (1-u)kh$. Given the linear expression for output, we can rewrite the Phillips curve as

$$\pi(u) = \pi^* + \frac{kh}{\varpi\delta(\delta - r)a} \cdot (1-u) \cdot \left[\frac{\tau(u)}{u} - 1\right]. \tag{H.9}$$

RESULT H.2. *The Phillips curve* $\pi(u)$ *given by* (H.9) *is strictly decreasing in* u *around the efficient unemployment rate* $u = u^*$ *and strictly convex in* u *on* $(\bar{u}, 1)$.

To establish the result, we differentiate (H.9) twice. The first derivative is

$$\pi'(u) = \frac{kh}{\varpi\delta(\delta - r)a} \cdot \left[(1-u) \cdot \frac{d\tau/u}{du} + 1 - \frac{\tau(u)}{u}\right].$$

One result that appears here is that $\pi'(u^*) < 0$. Indeed, we know from (9.11) and (H.2) that at efficiency, $\tau(u^*)/u^* = 1$. Hence, the sign of $\pi'(u^*)$ is the sign of $(1-u^*)d(\tau/u)/du$. But we know from result H.1 that $d(\tau/u)/du < 0$, so we see that $\pi'(u^*) < 0$. Hence, in the vicinity of efficiency, the Phillips curve is strictly decreasing in the unemployment rate.

Differentiating again, we obtain

$$\pi''(u) = \frac{kh}{\varpi\delta(\delta - r)a} \cdot \left[(1-u) \cdot \frac{d^2\tau/u}{du^2} - 2 \cdot \frac{d\tau/u}{du}\right].$$

From result H.1, we know that $d(\tau/u)/du < 0$ and that $d^2(\tau/u)/du^2 > 0$. Both terms in the square brackets are strictly positive, so $\pi''(u) > 0$ on $(\bar{u}, 1)$. Hence, the Phillips curve is strictly convex in the unemployment rate.