A Theory of Slack

How Economic Slack Shapes Markets, Business Cycles, and Policies

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APPENDIX F.

Mathematical background

This final appendix lists background mathematical results that we use repeatedly throughout the book.[1] These results are well known. They are presented, proved, and discussed in other texts, so we do not rederive them but simply collect them here for convenience. A complete treatment of the results is provided in the following textbooks: Apostol (1967) for calculus; Boyd and Vandenberghe (2004) for convexity of functions and convex optimization; Acemoglu (2009) for optimal control; and Hirsch, Smale, and Devaney (2013) for differential equations and dynamical systems.

Throughout the appendix, just like in the book, we use the following terminology. For numbers, "positive" means $x \geq 0$, "strictly positive" means $x > 0$, "negative" means $x \leq 0$, and "strictly negative" means $x < 0$. For functions, "increasing" means $x_1 < x_2 \Rightarrow f(x_1) \leq f(x_2)$ for all x_1, x_2 on the domain, "strictly increasing" means $x_1 < x_2 \Rightarrow f(x_1) < f(x_2)$, "decreasing" means $x_1 < x_2 \Rightarrow f(x_1) \geq f(x_2)$, and "strictly decreasing" means $x_1 < x_2 \Rightarrow f(x_1) > f(x_2)$. For dynamical systems, an "equilibrium point" means $\dot{x}(t) = 0$.

F.1. Miscellaneous results

RESULT F.1 (Intermediate-value theorem). *Consider a function f that is continuous on $[a, b]$. If $f(a)$ and $f(b)$ have opposite signs ($f(a)f(b) < 0$), then there exists $c \in (a, b)$ such*

[1] The book assumes only basic familiarity with common functions (power, exponential, logarithmic, and so on), differential calculus (derivatives, linearization, implicit differentiation, and so on), and introductory probability and statistics (least-squares regressions, normal distribution, and so on). This appendix does not repeat these fundamentals; it collects results that are slightly more advanced.

that $f(c) = 0$. If, in addition, f is strictly monotone on $[a, b]$, then this zero is unique.

RESULT F.2 (Euler's theorem for homogeneous functions). *Consider a differentiable function $f : (0, +\infty)^m \to (0, +\infty)$ that is homogeneous of degree $n > 0$, so that for all $x \in (0, +\infty)^m$ and $z > 0$:*

$$f(z \cdot x) = z^n \cdot f(x).$$

Then the function is related to its partial derivatives as follows:

$$f(x) = \sum_{i=1}^{m} \frac{x_i}{n} \cdot \frac{\partial f}{\partial x_i}.$$

In the case of a function that has constant returns to scale (homogeneous of degree 1), the relation simplifies to

$$f(x) = \sum_{i=1}^{m} x_i \cdot \frac{\partial f}{\partial x_i}.$$

This result implies that the partial elasticities of f add up to 1:

$$\sum_{i=1}^{m} \epsilon_{x_i}^{f} = \sum_{i=1}^{m} \frac{x_i}{f(x)} \cdot \frac{\partial f}{\partial x_i} = 1.$$

RESULT F.3 (Poisson process). *A counting process $\{N(t), t \geq 0\}$ is a Poisson process with rate $\lambda > 0$ if $N(0) = 0$, and for any $t > 0$ and $h > 0$, the increments $N(t+h) - N(t)$ are independent over disjoint time intervals and they are Poisson distributed with parameter λh:*

$$\mathbb{P}(N(t+h) - N(t) = k) = \frac{\exp(-\lambda h) \cdot (\lambda h)^k}{k!},$$

for any $k = 0, 1, 2, \ldots$. This implies that over a short time interval dt:

$$\mathbb{P}(N(t+dt) - N(t) = 1) = \lambda\, dt + o(dt), \qquad \mathbb{P}(N(t+dt) - N(t) > 1) = o(dt).$$

Thus, the probability that one event occurs in the near future is always just $\lambda\, dt$. Furthermore, the waiting time to the next event, T, has an exponential distribution with parameter λ:

$$\mathbb{P}(T < t) = 1 - \exp(-\lambda t), \qquad \mathbb{P}(T > t) = \exp(-\lambda t).$$

This implies in particular that the waiting time to the next event is memoryless:

$$\mathbb{P}(T > s + t \mid T > s) = \mathbb{P}(T > t).$$

It also implies that the mean waiting time simply is:

$$\mathbb{E}(T) = \frac{1}{\lambda}.$$

RESULT F.4 (Spectral identities). *For any square real matrix, the trace is equal to the sum of its eigenvalues and the determinant is equal to the product of its eigenvalues, counting repeated eigenvalues. If the matrix is symmetric, all its eigenvalues are real. In general, a real matrix may have complex eigenvalues, but nonreal eigenvalues occur in conjugate pairs, so trace and determinant remain real.*

RESULT F.5 (First-order recursions). *Consider the first-order recursion*

$$x_{t+1} = f(x_t).$$

A fixed point of the recursion x^* satisfies $x^* = f(x^*)$.

- *Convergence of the sequence $\{x(t)\}$ is not guaranteed. Depending on f and on the starting value, trajectories can converge, cycle, or diverge.*
- *If the sequence $\{x(t)\}$ is monotone and bounded, then it converges. Any limit point must be a fixed point x^*.*
- *If f is differentiable at x^*, there is local convergence if $|f'(x^*)| < 1$ and local divergence if $|f'(x^*)| > 1$.*

RESULT F.6 (Leibniz's integral rule). *Consider the integral*

$$I(z) = \int_{a(z)}^{b(z)} f(x,z)\,dx,$$

where the functions a and b are differentiable and the function f is continuously differentiable. Then the integral can be differentiated as follows:

$$\frac{dI}{dz} = \int_{a(z)}^{b(z)} \frac{\partial f}{\partial z}\,dx + \frac{db}{dz} f(b(z),z) - \frac{da}{dz} f(a(z),z).$$

F.2. The exponential and logarithm functions

RESULT F.7. *The exponential function is strictly convex, so it is above all its tangent, and in particular above the tangent at $x = 0$:*

$$\exp(x) \geq 1 + x$$

for all $x \in \mathbb{R}$, with equality only at $x = 0$.

RESULT F.8. *The logarithm function is strictly concave, so it is below all its tangent, and in particular below the tangent at $x = 1$:*

$$\ln(x) \leq x - 1,$$

for all $x > 0$, with equality only at $x = 1$.

RESULT F.9. *The exponential function equals its Taylor series at 0 (Maclaurin series):*

$$\exp(x) = \sum_{n=0}^{\infty} \frac{x^n}{n!}$$

for all $x \in \mathbb{R}$. This is an exact identity, not a local approximation: because the exponential function is analytic on $\mathbb{R}$, this series converges to the function everywhere.

F.3. Rational functions

RESULT F.10. *Consider a linear-over-linear rational function:*

$$R(x) = \frac{ax + b}{cx + d},$$

with $ad - bc \neq 0$ and $c \neq 0$. The function admits a single pole at $x = -d/c$, where $R(x) \to \pm\infty$, and a horizontal asymptote at $y = a/c$, when $x \to \pm\infty$. If $a \neq 0$, the function admits a single root at $x = -b/a$, where $R(x) = 0$. If $a = 0$, the function has no roots and its horizontal asymptote is $y = 0$. If $ad - bc > 0$, the function is strictly increasing and strictly convex on $(-\infty, -d/c)$ and strictly increasing and strictly concave on $(-d/c, +\infty)$. If $ad - bc < 0$, the function is strictly decreasing and strictly concave on $(-\infty, -d/c)$ and strictly decreasing and strictly convex on $(-d/c, +\infty)$.

RESULT F.11. *Consider the generic rational function*

$$R(x) = \frac{P(x)}{Q(x)},$$

where P and Q are polynomials with no common factor. Many of the properties of the function can be determined by graphing it:

- *Each real root of Q is a pole of R, where it presents a vertical asymptote.*
- *Each real root of P is a root of R, where it comes in contact with the x-axis:*
  - *If the root has multiplicity m and m is odd, the graph crosses the x-axis.*
  - *If the root has multiplicity m and m is even, the graph touches the x-axis.*

- *The limits of R at infinity are determined by polynomial degrees:*
  - *If* $\deg P < \deg Q$*, then* $\lim_{x\to\pm\infty} R(x) = 0$.
  - *If* $\deg P = \deg Q$*, the limit is the ratio of leading coefficients.*
  - *If* $\deg P = \deg Q + 1$*, there is an oblique asymptote obtained by polynomial division.*
  - *If* $\deg P > \deg Q+1$*, there is a polynomial asymptote obtained by polynomial division.*

F.4. Convexity of functions

The results below are borrowed from Boyd and Vandenberghe (2004, chapter 3).

RESULT F.12. *The function* f *is concave iff the function* $-f$ *is convex.*

From this result, we can infer all concavity result below from the convexity results. Nevertheless, to improve user-friendliness, we state both the convex and the concave cases.

RESULT F.13. *For a twice-differentiable function of one variable,* $f(x)$*, defined on a convex set, concavity and convexity can be checked from its second derivative:*

- *The function is concave if* $f''(x) \leq 0$ *on the set.*
- *The function is strictly concave if* $f''(x) < 0$ *on the set.*
- *The function is convex if* $f''(x) \geq 0$ *on the set.*
- *The function is strictly convex if* $f''(x) > 0$ *on the set.*

RESULT F.14. *For a twice-differentiable function of two variables,* $f(x, y)$*, defined on a convex set, concavity and convexity can be checked from its Hessian:*

$$H = \begin{bmatrix} \frac{\partial^2 f}{\partial x^2} & \frac{\partial^2 f}{\partial x \partial y} \\ \frac{\partial^2 f}{\partial y \partial x} & \frac{\partial^2 f}{\partial y^2} \end{bmatrix}.$$

- *The function is concave iff the Hessian is negative semidefinite on the set. This requires that the Hessian's two, real eigenvalues are negative, and in turn that its determinant is positive and is trace negative.*
- *The function is strictly concave if the Hessian is negative definite. This requires that the Hessian's two, real eigenvalues are strictly negative, and in turn that its determinant is strictly positive and is trace strictly negative.*
- *The function is convex iff the Hessian is positive semidefinite on the set. This requires that the Hessian's two, real eigenvalues are positive, and in turn that its determinant is positive and is trace positive.*

- *The function is strictly convex if the Hessian is positive definite. This requires that the Hessian's two, real eigenvalues are strictly positive, and in turn that its determinant is strictly positive and is trace strictly positive.*

RESULT F.15. *Consider an affine function $g(x) = ax + b$ with $a \neq 0$.*

- *If f is (strictly) convex, then $f \circ g$ is (strictly) convex.*
- *If f is (strictly) concave, then $f \circ g$ is (strictly) concave.*
- *If f is convex and $a \geq 0$, then $g \circ f$ is convex.*
- *If f is concave and $a \geq 0$, then $g \circ f$ is concave.*
- *If f is strictly convex and $a > 0$, then $g \circ f$ is strictly convex.*
- *If f is strictly concave and $a > 0$, then $g \circ f$ is strictly concave.*

RESULT F.16. *The positive weighted sum of convex functions is convex. If any term in the sum is a strictly convex function with a strictly positive weight, the sum is strictly convex. Similarly, the positive weighted sum of concave functions is concave. If any term in the sum is a strictly concave function with a strictly positive weight, the sum is strictly concave.*

RESULT F.17. *Consider first two twice-differentiable functions $f : \mathbb{R} \to \mathbb{R}$ and $g : \mathbb{R} \to \mathbb{R}$. The composition $f \circ g$ of the two functions has the following properties:*

- *If f is convex and increasing, and g is convex, then $f \circ g$ is convex.*
- *If f is convex and decreasing, and g is concave, then $f \circ g$ is convex.*
- *If f is concave and increasing, and g is concave, then $f \circ g$ is concave.*
- *If f is concave and decreasing, and g is convex, then $f \circ g$ is concave.*

If the two functions are not defined on $\mathbb{R}$ but on some convex subset of $\mathbb{R}$, then the results continue to hold once we add a constraint on the domain of f:

- *If f is convex, f is increasing on an interval of the form $(-\infty, a)$, and g is convex, then $f \circ g$ is convex.*
- *If f is convex, f is decreasing on an interval of the form $(a, +\infty)$, and g is concave, then $f \circ g$ is convex.*
- *If f is concave, f is increasing on an interval of the form $(a, +\infty)$, and g is concave, then $f \circ g$ is concave.*
- *If f is concave, f is decreasing on an interval of the form $(-\infty, a)$, and g is convex, then $f \circ g$ is concave.*

RESULT F.18. *Consider a twice-differentiable bijection f. Then its inverse has the following* *properties:*

- *If f is strictly increasing and strictly convex, then its inverse is strictly increasing and strictly concave.*
- *If f is strictly decreasing and strictly convex, then its inverse is strictly decreasing and strictly convex.*
- *If f is strictly increasing and strictly concave, then its inverse is strictly increasing and strictly convex.*
- *If f is strictly decreasing and strictly concave, then its inverse is strictly decreasing and strictly concave.*

PROOF. By the inverse function theorem, its inverse has first derivative

$$(f^{-1})'(y) = \frac{1}{f'(f^{-1}(y))}$$

and second derivative

$$(f^{-1})''(y) = -\frac{1}{\left[f'(f^{-1}(y))\right]^2} \cdot f''(f^{-1}(y)) \cdot (f^{-1})'(y) = -\frac{f''(f^{-1}(y))}{\left[f'(f^{-1}(y))\right]^3}.$$

We see that the inverse's first derivative $(f^{-1})'$ has the same sign as the function's first derivative f'. We also see that the inverse's second derivative $(f^{-1})''$ has the same sign as the function's second derivative f'' if the function is decreasing ($f' < 0$), but opposite sign if the function is increasing ($f' > 0$). All the results follow from these observations. □

F.5. Convex optimization

Convex optimization is concerned with static optimization of convex (or concave) functions over convex sets.

RESULT F.19. *If an objective function is strictly convex on a convex domain, then any interior point satisfying the first-order condition is the unique global minimum.*

RESULT F.20. *If an objective function is strictly concave on a convex domain, then any interior point satisfying the first-order condition is the unique global maximum.*

F.6. Optimal control

Optimal control is concerned with dynamic optimization in continuous time. This section's results below are borrowed from Acemoglu (2009, chapter 7).

RESULT F.21. *Consider the following optimal control problem:*

$$\max_{\{c(t)\}_{t\geq 0}} \int_0^\infty e^{-\delta t} u(a(t), c(t))\, dt$$

subject to the law of motion

$$\dot{a}(t) = g\left(a(t), c(t)\right).$$

Introduce a costate $q(t)$ for the state equation. Then the current-value Hamiltonian is

$$\mathcal{H}(t) = u(a(t), c(t)) + q(t) \cdot g(a(t), c(t)).$$

With several state equations, add one costate term for each law of motion.

RESULT F.22. *For an optimal control problem with current-value Hamiltonian $\mathcal{H}$, control c, state a, and costate q, interior optima satisfy*

$$\frac{\partial \mathcal{H}}{\partial c} = 0$$
$$\frac{\partial \mathcal{H}}{\partial a} = \delta q - \dot{q},$$

with an appropriate transversality condition. A typical transversality condition is

$$\lim_{t\to\infty} \exp(-\delta t) q(t) a(t) = 0.$$

The transversality condition rules out explosive paths that violate optimality. In practice it is used to eliminate trajectories that diverge and to select the economically relevant path.

F.7. Differential equations and dynamical systems

Differential equations are functional equations in continuous time, and dynamical systems are collections of differential equations. This section's results below are borrowed from Hirsch, Smale, and Devaney (2013, chapters 1–6).

Differential equations

RESULT F.23. *The exponential function is defined as the only solution to the differential equation*

$$\dot{x}(t) - x(t) = 0,$$

with initial condition $x(0) = 1$. Hence, the unique solution to differential equations

$$\dot{x}(t) - \lambda x(t) = 0,$$

with initial condition $x(t_0) = x_0$, is

$$x(t) = x_0 \cdot \exp(\lambda\,(t - t_0)).$$

RESULT F.24. *Consider the linear first-order differential equation*

$$\dot{x}(t) - \lambda(t)x(t) = f(t).$$

Its solution with initial condition $x(t_0) = x_0$ is

$$x(t) = x_0 \exp\left(\int_{t_0}^{t} \lambda(s)\,ds\right) + \int_{t_0}^{t} f(z) \exp\left(\int_{z}^{t} \lambda(s)\,ds\right) dz.$$

F.7.1. Dynamical systems

RESULT F.25. *For a linear system $\dot{x} = Ax$, with $x \in \mathbb{R}^n$ and $A \in \mathbb{R}^{n\times n}$, the local dynamics around an equilibrium point are governed by eigenvalues of A. If A is diagonalizable with eigenpairs (λ_i, z_i), then the homogeneous solution can be written*

$$x(t) = \sum_{i=1}^{n} A_i z_i \exp(\lambda_i t),$$

where constants A_i are pinned down by boundary or initial conditions.

F.7.2. Phase diagrams

RESULT F.26. *In two-dimensional systems, nullclines $\dot{x} = 0$ and $\dot{y} = 0$ partition the plane. The sign of $\dot{x}$ and $\dot{y}$ in each region determines the local direction of motion.*

RESULT F.27. *Linearization around an equilibrium point replaces a nonlinear system with the Jacobian system. Local properties (source, sink, saddle) are preserved in a neighborhood of the equilibrium.*

RESULT F.28. *With one predetermined state variable and one jump control variable in a saddle system, the control jumps on impact to place the economy on the stable path.*

RESULT F.29. *For a 2×2 linear system, the discriminant*

$$\Delta = \mathrm{tr}(A)^2 - 4\det(A)$$

distinguishes real and complex eigenvalues and this dynamic types: $\Delta > 0$ gives real eigenvalues and nodal dynamics, while $\Delta < 0$ gives complex eigenvalues and spiral dynamics. Furthermore:

- *If* $\det(A) < 0$*, eigenvalues have opposite signs, so the equilibrium is a saddle.*
- *If* $\det(A) > 0$ *and* $\operatorname{tr}(A) < 0$*, both eigenvalues have negative real parts, so the equilibrium is a sink.*
- *If* $\det(A) > 0$ *and* $\operatorname{tr}(A) > 0$*, both eigenvalues have positive real parts, so the equilibrium is a source.*

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