A Theory of Slack

How Economic Slack Shapes Markets, Business Cycles, and Policies

Pascal Michaillat

Draft version: December 2025

APPENDIX B.

Elasticities

In this appendix, we review the concept of elasticity, which we use repeatedly throughout the book. Throughout we consider a function $f(x)$ that is positive and differentiable for $x > 0$.

B.1. Definition of elasticities

The elasticity of $f(x)$ with respect to x, denoted ϵ_x^f, is defined as

$$\epsilon_x^f = \frac{x}{f(x)} \cdot f'(x), \tag{B.1}$$

where $f'(x)$ is the standard derivative of f. All properties of elasticities flow from this single definition.

B.2. Leibniz notation for differentials

To work with elasticities, it is convenient to introduce Leibniz's notation for differentials. In this notation, dx denotes a small, infinitesimal change in a variable x, and $df(x)$ denotes a small, infinitesimal change in a function $f(x)$. The differential $df(x)$ can be computed from the differential dx and the derivative $f'(x)$ of the function f:

$$df(x) = f'(x)\, dx. \tag{B.2}$$

The Leibniz notation is particularly convenient to use with the logarithm. Given that the derivative of $\ln(x)$ is $1/x$:

$$d\ln(x) = \frac{dx}{x}. \tag{B.3}$$

Hence, while dx measures the absolute change in x, $d\ln(x)$ measures a relative change in x.

Applying the logarithm to the function f, we obtain

$$d\ln f(x) = \frac{df(x)}{f(x)}. \tag{B.4}$$

So while $df(x)$ measures the absolute change in $f(x)$, $d\ln f(x)$ measures a relative change in $f(x)$.

These equations show that the log-differentials $d\ln(x)$ and $d\ln f(x)$ are simply a compact way of writing the relative change in the variable x or the function $f(x)$.

From definition (B.2), we see that it is easy to use the differential operator d because the standard laws of derivatives apply to it.

B.3. Interpretation of elasticities

What does an elasticity mean? To answer this, we rewrite the definition (B.1) using the Leibniz notation $f'(x) = df/dx$:

$$\epsilon_x^f = \frac{x}{f(x)} \cdot \frac{df(x)}{dx} = \frac{df(x)/f(x)}{dx/x}.$$

We see from this equation that the elasticity is the ratio of the relative change in f to the relative change in x. This gives a simple interpretation. Let's consider a 1% change in x, which means its relative change is $dx/x = 0.01$. The resulting relative change in $f(x)$ is:

$$\frac{df(x)}{f(x)} = \epsilon_x^f \times \frac{dx}{x} = \epsilon_x^f \times 0.01.$$

Thus, a 1% change in x leads to an ϵ_x^f percentage change in $f(x)$: the elasticity measures the percentage response of a function to a 1% change in its input.

B.4. Connection between elasticities and log-differentials

The reason elasticities are so convenient is their connection to natural logarithms. This connection comes from a basic result in calculus: the derivative of $\ln(x)$ is $1/x$. In Leibniz's notation, this means $d\ln(x) = dx/x$, as we saw in equation (B.3). Using this insight and expression (B.3), we can express the elasticity as the ratio of two log-differentials. This is not a new definition, but a convenient reformulation of (B.1):

$$\epsilon_x^f = \frac{d\ln f(x)}{d\ln(x)}. \tag{B.5}$$

This logarithmic form often helps compute the elasticity of complex functions.

Reshuffling (B.5), we also see that the elasticity relates the log-differential of the function to the log-differential of the argument:

$$d\ln f(x) = \epsilon_x^f \, d\ln(x).$$

B.5. Useful results on elasticities

Finally, we introduce several results that we use throughout the book to compute elasticities. Throughout, the functions $g(x)$ and $h(x)$ are assumed to be positive and differentiable for $x > 0$, and the parameter a is assumed to be positive.

RESULT 1. *The elasticity of the function* $f(x) = x^a$ *is* $\epsilon_x^f = a$.

PROOF. Since $\ln(f(x)) = a\ln(x)$, differentiating gives $d\ln(f(x)) = a\,d\ln(x)$. Dividing by $d\ln(x)$, we get $d\ln(f(x))/d\ln(x) = a$. We then obtain the result by applying (B.5). □

RESULT 2. *The elasticity of the function* $f(x) = ag(x)$ *is* $\epsilon_x^f = \epsilon_x^g$.

PROOF. Since $\ln(f(x)) = \ln(a) + \ln(g(x))$, and $\ln(a)$ is just a constant, then differentiating gives $d\ln(f(x)) = d\ln(g(x))$, so $d\ln(f(x))/d\ln(x) = d\ln(g(x))/d\ln(x)$. Once again, we obtain the result by applying (B.5). □

RESULT 3. *The elasticity of the function* $f(x) = g(x) \cdot h(x)$ *is* $\epsilon_x^f = \epsilon_x^g + \epsilon_x^h$.

PROOF. Since $\ln(f(x)) = \ln(g(x)) + \ln(h(x))$, then $d\ln(f(x)) = d\ln(g(x)) + d\ln(h(x))$. We obtain the result after dividing by $d\ln(x)$ and using (B.5). □

RESULT 4. *The elasticity of the function* $f(x) = g(x)/h(x)$ *is* $\epsilon_x^f = \epsilon_x^g - \epsilon_x^h$.

PROOF. Since $\ln(f(x)) = \ln(g(x)) - \ln(h(x))$, then $d\ln(f(x)) = d\ln(g(x)) - d\ln(h(x))$. Again, we obtain the result after dividing by $d\ln(x)$ and using (B.5). □

RESULT 5. *The elasticity of the function* $f(x) = g(x) + h(x)$ *is*

$$\epsilon_x^f = \frac{g(x)}{g(x) + h(x)} \cdot \epsilon_x^g + \frac{h(x)}{g(x) + h(x)} \cdot \epsilon_x^h.$$

PROOF. For sums of functions, the log definition (B.5) is not helpful, so we return to the original definition (B.1). Since $f'(x) = g'(x) + h'(x)$, we get

$$\begin{aligned}
\epsilon_x^f &= \frac{x}{f(x)} \cdot f'(x) \\
&= \frac{x}{g(x)+h(x)} \left[g'(x) + h'(x)\right] \\
&= \frac{g(x)}{g(x)+h(x)} \left[\frac{x}{g(x)} \cdot g'(x)\right] + \frac{h(x)}{g(x)+h(x)} \left[\frac{x}{h(x)} \cdot h'(x)\right],
\end{aligned}$$

which yields the result using the definition of elasticities, (B.1). □

RESULT 6. *The elasticity of the function $f(x) = g(x) - h(x)$, with $h(x) < g(x)$, is*

$$\epsilon_x^f = \frac{g(x)}{g(x)-h(x)} \cdot \epsilon_x^g - \frac{h(x)}{g(x)-h(x)} \cdot \epsilon_x^h.$$

PROOF. Here again, we use the original definition of the elasticity, (B.1). Since $f'(x) = g'(x) - h'(x)$, we get

$$\begin{aligned}
\epsilon_x^f &= \frac{x}{f(x)} \cdot f'(x) \\
&= \frac{x}{g(x)-h(x)} \left[g'(x) - h'(x)\right] \\
&= \frac{g(x)}{g(x)-h(x)} \left[\frac{x}{g(x)} \cdot g'(x)\right] - \frac{h(x)}{g(x)-h(x)} \left[\frac{x}{h(x)} \cdot h'(x)\right],
\end{aligned}$$

which yields the result using the definition of elasticities, (B.1). □

RESULT 7. *The elasticity of the function $f(x) = g(h(x))$ is*

$$\epsilon_x^f = \epsilon_x^h \cdot \epsilon_h^g,$$

where ϵ_h^g is the elasticity of g evaluated at $h(x)$.

PROOF. We again use the original definition of the elasticity, given by (B.1). By the chain rule $f'(x) = h'(x)g'(h(x))$, so we get

$$\begin{aligned}
\epsilon_x^f &= \frac{x}{f(x)} \cdot f'(x) \\
&= \left[\frac{x}{h(x)} \cdot h'(x)\right]\left[\frac{h(x)}{g(h(x))} \cdot g'(h(x))\right],
\end{aligned}$$

which yields the result using the definition of elasticities, (B.1). □

Finally, we introduce a bivariate function $k(y, z)$ which is positive and differentiable for $(y, z) \in (0, \infty)^2$. We also assume that the arguments y and z are themselves functions of x, and the functions $y(x)$ and $z(x)$ are positive and differentiable for $x > 0$.

RESULT 8. *The elasticity of the function* $f(x) = k(y(x), z(x))$ *is*

$$\epsilon_x^f = \epsilon_y^k \cdot \epsilon_x^y + \epsilon_z^k \cdot \epsilon_x^z,$$

where ϵ_y^k *and* ϵ_z^k *are partial elasticities of the function* k*:*

$$\epsilon_y^k = \frac{y}{k(y,z)} \cdot \frac{\partial k}{\partial y} \qquad \text{and} \qquad \epsilon_z^k = \frac{z}{k(y,z)} \cdot \frac{\partial k}{\partial z}.$$

PROOF. We once more use the original definition of the elasticity, given by (B.1). By the multivariate chain rule,

$$f'(x) = \frac{\partial k}{\partial y} \cdot y'(x) + \frac{\partial k}{\partial z} \cdot z'(x).$$

Hence, using the fact that $f(x) = k(y(x), z(x))$, we get

$$\begin{aligned}
\epsilon_x^f &= \frac{x}{f(x)} \cdot f'(x) \\
&= \left[\frac{y}{k(y,z)} \cdot \frac{\partial k}{\partial y}\right]\left[\frac{x}{y} \cdot y'(x)\right] + \left[\frac{z}{k(y,z)} \cdot \frac{\partial k}{\partial z}\right]\left[\frac{x}{z} \cdot z'(x)\right].
\end{aligned}$$

This equation yields the result using the definition of the standard and partial elasticities. □

RESULT 9. *Consider two variables* y *and* x*, related by* $y = f(x)$*, where* f *is a bijection with* $f'(x) \neq 0$*. Let* $g = f^{-1}$ *be the inverse of* f*; it is differentiable by the inverse function theorem, and* $x = g(y)$*. The elasticity of the function* g *with respect to* y *is*

$$\epsilon_y^g = \frac{1}{\epsilon_x^f},$$

where ϵ_x^f *is the elasticity of* f *evaluated at* $x = g(y)$*.*

PROOF. Once again, we use the original definition of the elasticity, given by (B.1). By the inverse function theorem $g'(y) = 1/f'(f^{-1}(y))$, so we get

$$\epsilon_y^g = \frac{y}{f^{-1}(y)} \cdot \frac{1}{f'(f^{-1}(y))}.$$

Using the definition $\epsilon_x^f = xf'(x)/f(x)$ and substituting $x = g(y) = f^{-1}(y)$, we obtain the result. □

B.6. Implicit differentiation with elasticities

Implicit differentiation is very helpful to compute derivatives of functions that are defined implicitly. It turns out that we can also use the same technique to compute elasticities of functions defined implicitly.

RESULT 10. *Assume that the function $y(x) > 0$ is defined implicitly by the equation $f(x, y) = g(x, y)$, where the bivariate functions $f(x, y)$ and $g(x, y)$ are positive and differentiable for $(x, y) \in (0, \infty)^2$. Then we can take elasticities on both sides of the equation so the elasticity of y with respect to x satisfies*

$$\epsilon_x^f + \epsilon_y^f \cdot \epsilon_x^y = \epsilon_x^g + \epsilon_y^g \cdot \epsilon_x^y,$$

where ϵ_x^f, ϵ_y^f, ϵ_x^g, ϵ_y^g, are partial elasticities defined by

$$\epsilon_x^f = \frac{x}{f(x,y)} \cdot \frac{\partial f}{\partial x}, \quad \epsilon_y^f = \frac{y}{f(x,y)} \cdot \frac{\partial f}{\partial y}, \quad \epsilon_x^g = \frac{x}{g(x,y)} \cdot \frac{\partial g}{\partial x}, \quad \epsilon_y^g = \frac{y}{g(x,y)} \cdot \frac{\partial g}{\partial y}.$$

Collecting terms, we can express the elasticity of y as a function of the elasticities of the functions f and g:

$$\epsilon_x^y = \frac{\epsilon_x^g - \epsilon_x^f}{\epsilon_y^f - \epsilon_y^g}.$$

PROOF. The implicit differentiation of the equation $f(x, y) = g(x, y)$ gives

$$\frac{\partial f}{\partial x} + \frac{\partial f}{\partial y} \cdot y'(x) = \frac{\partial g}{\partial x} + \frac{\partial g}{\partial y} \cdot y'(x).$$

We then divide the left-hand side by $f(x, y)$ and the right-hand side by $g(x, y)$, which we can do because they have the same value, and we multiply both sides by x. We get

$$\frac{x}{f(x,y)} \cdot \frac{\partial f}{\partial x} + \left[\frac{y}{f(x,y)} \cdot \frac{\partial f}{\partial y}\right]\left[\frac{x}{y} \cdot y'(x)\right] = \frac{x}{f(x,y)} \cdot \frac{\partial g}{\partial x} + \left[\frac{y}{f(x,y)} \cdot \frac{\partial g}{\partial y}\right]\left[\frac{x}{y} \cdot y'(x)\right],$$

which yields the result using the definition of the standard and partial elasticities. □